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Calculation of cocyclic matrices

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Abstract

In this paper we provide a method of explicitly determining, for a given finite group G and finitely generated G-module U trivial under the action of G, a representative for each element (2-cocycle class) in $H^2(G,U)$. These cocycles are naturally displayed as $|G| \times |G|$ matrices. An example of calculating cocyclic matrices using the method is given.

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1. Introduction

Let G be a finite group and U a G-module. For each $n \geq 0$ one may define the familiar nth cohomology group $H^n(G,U)$ of G with coefficients in U, the elements of which are n-cocycle classes. The problem of explicitly determining a full set of representative n-cocycles for given G and U does not appear to have been traditionally studied by cohomologists, although the need for such information has arisen in several areas. One example involves an application in combinatorial design theory. A 2-cocycle ψ is naturally displayed as a *cocyclic matrix* (*associated with* ψ, *developed over* G); that is, a square matrix whose rows and columns are indexed by the elements of G (under some fixed ordering) and whose entry in position (g,h) is $\psi(g,h)$. This notion was used in recent work by Horadam and de Launey [3] as a modification of group development of designs, in part as a new way of generating designs. It is also apparent that cocyclic matrices, associated with cocycles with coefficients in $\mathbb{Z}_2 = \{-1,1\}$, account for large classes of so-called Hadamard matrices, and may consequently provide a uniform approach to the famous Hadamard conjecture. In this

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context, methods for calculating cocyclic matrices are required, thus motivating the main problem considered in this paper. (Note that our focus is entirely on construction of representative cocycles. The construction of 2-coboundaries is straightforward from the definition, and best relegated to a computer, given the number of such objects involved. For example, when $U = \mathbb{Z}_2$, the number of 2-coboundaries $|B^2(G,U)|$ grows as $2^{|G|-r-1}$, where r is the rank of the Sylow 2-subgroup of G/G'.)

Our method is based on an explicit version of the well-known Universal Coefficient Theorem, which provides a decomposition of the second cohomology group into the direct sum of two summands. The summands may be calculated as the images of certain embeddings, called inflation and transgression. These homomorphisms arise in exact sequences derived from the Lyndon–Hochschild–Serre spectral sequence. We quote the relevant results in this area in Section 2 of the paper. The specific Universal Coefficient Theorem decomposition is then discussed in Section 3. In Section 4, we examine symmetry properties of cocycles produced by the method. Finally, in Section 5 we provide an example of calculating cocyclic matrices.

Familiarity with standard concepts and notation in the cohomology theory of groups, as may be found in Ch. VI of [2], will be assumed. Throughout, we consider U as a left G-module, and as a multiplicative abelian group unless stated otherwise. For "2-cocycles" it is often convenient to write "cocycles". Cohomology class is denoted $[-]$. All cocycles considered are normalised.

2. Inflation, restriction and transgression

References for the material in this section are Section 10 of [7] and pp. 38–52 of [5].

Let N be a normal subgroup of G and denote by U^N the G-submodule of N-fixed points of U. For each $\psi \in Z^n(G/N, U^N)$, define $\inf\psi \in Z^n(G,U)$ by

$$\inf\psi(g_1,\dots,g_n) = \psi(g_1N,\dots,g_nN).$$

Setting $\inf[\psi] = [\inf\psi]$ defines inflation from $H^n(G/N, U^N)$ to $H^n(G, U^N)$. Restriction is the homomorphism res : $H^n(G,U) \to H^n(N,U)$ induced by restricting to N the domain of definition of n-cocycles.

Inflation on second cohomology has a felicitous description in terms of cocyclic matrices. Choose $\psi \in Z^2(G/N, U^N)$, label the elements of N as $x_1 = 1, x_2, \dots, x_s$ and choose a set of representatives $g_1 = 1, g_2, \dots, g_r$ for the cosets G/N. Denote by $M_{\inf\psi}$ and M_ψ the cocyclic matrices associated with $\inf\psi$ and ψ, where rows and columns are indexed $1, g_2, \dots, g_r, x_2, \dots, g_rx_2, \dots, x_s, \dots, g_rx_s$ and $N, g_2N, \dots, g_rN$, respectively. Clearly,

$$M_{\inf\psi} = M_\psi \otimes J_s,$$

where $\otimes$ denotes Kronecker product of matrices and J_s is the $s \times s$ all 1s matrix. Informally, inflation is just "tensoring up".

Suppose for the rest of this section that N is a central subgroup of G and G acts trivially on U. Transgression $\tau : \mathrm{Hom}(N, U) \to H^2(G/N, U)$ is defined as follows. By assumption,

$$1 \to N \xrightarrow{\text{inc.}} G \xrightarrow{\text{proj.}} G/N \to 1$$

is a central extension of N by G/N, and so we obtain in the usual way an associated cocycle $\mu_\sigma \in Z^2(G/N, N)$ defined via a normalised transversal function $\sigma : G/N \to G$. The assignment

$$\tau : \phi \mapsto [\phi \circ \mu_\sigma]$$

is a homomorphism from $\mathrm{Hom}(N, U)$ into $H^2(G/N, U)$. Of course, each $\mu \in Z^2(G/N, N)$ gives rise to a homomorphism $\phi \mapsto [\phi \circ \mu]$ from $\mathrm{Hom}(N, U)$ to $H^2(G/N, U)$. However, unlike τ, such a homomorphism does not necessarily possess the same kernel and image as the Lyndon–Hochschild–Serre spectral sequence differential $d_2^{0,1}$. This property allows us to write $\tau = d_2^{0,1}$ in the place of $d_2^{0,1}$ in the following "fundamental" five term exact sequence (see [7, p. 354, Eq. (10.6); and 5, p. 45, Theorem 2.5]):

$$\begin{aligned} 0 &\to \mathrm{Hom}(G/N, U) \xrightarrow{\inf} \mathrm{Hom}(G, U) \xrightarrow{\text{res}} \mathrm{Hom}(N, U) \\ &\xrightarrow{\tau} H^2(G/N, U) \xrightarrow{\inf} H^2(G, U). \end{aligned} \tag{1}$$

The next result is a useful consequence of (1). Write $g^{-1}h^{-1}gh = [g, h]$ for $g, h \in G$.

Lemma 2.1. *Suppose that G, N, U are as above and also that $N \leq G'$. Then τ : $\mathrm{Hom}(N, U) \to H^2(G/N, U)$ is injective.*

Proof. For any $\theta \in \mathrm{Hom}(G, U)$ and $g, h \in G$, $\theta([g, h]) = [\theta(g), \theta(h)] = 1$. Hence res : $\mathrm{Hom}(G, U) \to \mathrm{Hom}(N, U)$ is zero, implying the result by (1). □

3. The Universal Coefficient Theorem and calculation of cocyclic matrices

In this section, we draw on several standard ideas, which are collected most conveniently in [6] (mainly Section 2.1 of that book).

Throughout, G is finite and U a trivial G-module. Denote the Schur multiplicator of G by $H_2(G)$. If $G = A$ is abelian then $\mathrm{Ext}(A, U)$ denotes the subgroup of $H^2(A, U)$ consisting of the 2-cocycle classes $[\psi]$ containing a *symmetric cocycle* ψ, meaning that $\psi(g, h) = \psi(h, g)$ for all $g, h \in G$. For the next result, see [2], p. 179, Theorem 3.3.

Theorem 3.1 (Universal Coefficient Theorem). *For G and U as above,*

$$H^2(G, U) \cong \mathrm{Ext}(G/G', U) \oplus \mathrm{Hom}(H_2(G), U). \tag{2}$$

Our immediate objective is to obtain the decomposition (2) as an internal direct sum. To accomplish this, at least in the case that U is finitely generated, we exhibit embeddings of $\mathrm{Ext}(G/G', U)$ and $\mathrm{Hom}(H_2(G), U)$ in $H^2(G, U)$ whose images are complementary.

There is an embedding of $\mathrm{Ext}(G/G', U)$ in $H^2(G, U)$, arising as the restriction to $\mathrm{Ext}(G/G', U)$ of the inflation homomorphism on $H^2(G/G', U)$. Inflation maps symmetric cocycles to symmetric cocycles, so that, in particular, representative cocyclic matrices for the elements of this embedding's image may be chosen as symmetric matrices. The fact that inflation is injective on $\mathrm{Ext}(G/G', U)$ is a consequence of the following technical proposition (this is formulated in a general context which allows the crucial Theorem 3.5 also to be treated as a special case).

Proposition 3.2. *Let K be a group acting trivially on U, and N a normal subgroup of K such that $K/N \cong G$ and $N \leq K'$. Denote by* $\inf_1$ *the restriction to* $\mathrm{Ext}(G/G', U)$ *of inflation*

$$H^2(G/G', U) \to H^2(G, U),$$

and by $\inf_2$ *inflation*

$$H^2(G, U) \to H^2(K, U).$$

Then the composite $\inf_2 \inf_1$ *is injective.*

Proof. Choose $[\psi] \in \mathrm{Ext}(G/G', U)$ such that $\inf_2 \inf_1 [\psi] = 0$. Since G/G' is naturally isomorphic to K/K', this implies that there is $[\tilde{\psi}] \in \mathrm{Ext}(K/K', U)$ corresponding to $[\psi]$ and a normalised 1-cochain $\phi : K \to U$ such that

$$\tilde{\psi}(xK', yK') = \phi(x)\phi(y)\phi(xy)^{-1} \tag{3}$$

for all $x, y \in K$. Hence,

$$\phi(xt) = \phi(x)\phi(t) \tag{4}$$

for all $t \in K'$, implying that

$$\phi(xy) = \phi(yx)\phi([x, y]).$$

But symmetry of $\tilde{\psi}$ and (3) force $\phi(xy) = \phi(yx)$, so that $\phi([x, y]) = 1$. By (4), $\phi|_{K'}$ is certainly a homomorphism, hence the identity homomorphism, and the normalised map φ from K/K' to U defined by $\varphi(xK') = \phi(x)$ is well-defined. Then $\tilde{\psi} \in B^2(K/K', U)$ by (3), verifying the proposition. □

From now on, unless otherwise stated, inf will denote the restriction to $\mathrm{Ext}(G/G', U)$ of the inflation homomorphism on $H^2(G/G', U)$.

Corollary 3.3. inf *is injective.*

Proof. Take $K = G$ and $N = 1$ in Proposition 3.2. □

We complete our description of inf with an outline of a procedure for determining a full set of representative cocyclic matrices for the elements of $\mathrm{Ext}(G/G', U)$. Write G/G' in primary invariant form $G/G' \cong \prod_i \mathbb{Z}_{p_i^{e_i}}$, where the e_i range over some finite set of positive integers and the p_i are primes. Since the bifunctor Ext is additive,

$$\mathrm{Ext}(G/G', U) \cong \bigoplus_i \mathrm{Ext}(\mathbb{Z}_{p_i^{e_i}}, U).$$

By selecting a representative cocyclic matrix for an element of each $\mathrm{Ext}(\mathbb{Z}_{p_i^{e_i}}, U)$, and forming the Kronecker product of these matrices, we obtain a representative cocyclic matrix for an element of $\mathrm{Ext}(G/G', U)$. The set of all such Kronecker products is a full set of representative cocyclic matrices for the elements of $\mathrm{Ext}(G/G', U)$. A representative cocyclic matrix for each element of $\mathrm{Ext}(\mathbb{Z}_{p_i^{e_i}}, U)$ is obtained by first noting that the latter group is precisely $H^2(\mathbb{Z}_{p_i^{e_i}}, U)$ – a central extension of an abelian group by a cyclic group is always abelian. The cohomology of finite cyclic groups is very well-known; in particular, there is an isomorphism $H^2(\mathbb{Z}_n, U) \cong U/U^n$ which may be explicitly defined (see [5, p. 52, Theorem 3.1]). For example, a representative cocyclic matrix for the single nonzero element of $H^2(\mathbb{Z}_{2n}, \mathbb{Z}_2) \cong \mathbb{Z}_2$ is the "back negacyclic" matrix of side $2n$, whose ith row is

$$1 \quad 1 \ldots 1 \quad -1 \quad -1 \ldots -1,$$

where the first occurrence of -1 is in column $2n - i + 2$.

The description of the embedding of $\mathrm{Hom}(H_2(G), U)$ in $H^2(G, U)$ makes use of Hopf's formula for $H_2(G)$. As is well-known, for any choice of presentation $G \cong F/R$, where F is free of finite rank, the finitely generated abelian group $R/[R,F]$ splits over its torsion subgroup $R \cap F'/[R,F] \cong H_2(G)$. A complement $S/[R,F]$ of $R \cap F'/[R,F]$ in $R/[R,F]$ (here called a *Schur complement*) is not necessarily unique, and its isomorphism type is dependent on the choice of presentation. Specifically, $S/[R,F]$ is free abelian of the same rank as F. Now note that

$$1 \to R/S \xrightarrow{\iota} F/S \xrightarrow{\pi} F/R \to 1, \tag{5}$$

where ι is inclusion and π the composite of projection and a natural isomorphism, is a central extension of $R/S \cong H_2(G)$ by $F/R \cong G$. As in Section 2, we may define transgression $\tau_S : \mathrm{Hom}(R/S, U) \to H^2(F/R, U)$.

Proposition 3.4. *τ_S is injective.*

Proof. Since $R/S \leq F'R/S = F'S/S = (F/S)'$, the result is a consequence of Lemma 2.1. □

Next, we show that the images of inflation and transgression are complementary.

Theorem 3.5. *Choose F, R and S as above. Then, identifying G with F/R,*

$$\mathrm{im}\,\mathrm{inf} \cap \mathrm{im}\,\tau_S = \{0\}.$$

Proof. Since $R/S \leq (F/S)'$, upon setting $K = F/S$ and $N = R/S$ the hypotheses of Proposition 3.2 are satisfied. In the present terms $\inf_1$ is inf, and by (1), $\inf_2$ has kernel $\operatorname{im} \tau_S$. The conclusion follows. □

This leads directly to the main result of the section (cf. [6, pp. 25–26, Theorem 2.1.19]).

Theorem 3.6. *Choose F, R and S as above, and let U be finitely generated. Then, identifying G with F/R and $H_2(G)$ with R/S,*

$$H^2(G, U) = \operatorname{im} \inf \oplus \operatorname{im} \tau_S.$$

Proof. Since $H^2(G, U)$ is finite in this situation, the result follows immediately from Corollary 3.3, Proposition 3.4, and Theorems 3.1 and 3.5. □

The theory in Section 4 of [1] and Sections 10–12 of [3] overlaps the theory presented in this section (as may be seen after choice of the standard presentation for G above). Theorem 3.6 supplies a general method for calculating a full set of representative cocycles for the elements of $H^2(G, U)$ when G is not necessarily abelian, lacking in [1,3].

To close this section, we note that *any* covering group of G and correspondingly defined transgression can take the role of F/S and τ_S, respectively, in Theorem 3.6. But this is of little import, since any covering group is isomorphic to F/S for some choice of S ([6, p. 50, Theorem 2.4.6 (iv) (b)]).

4. Symmetry of cocyclic matrices

Calculation of representative cocyclic matrices associated with elements of $\operatorname{im} \inf$ is canonical in the sense that once a presentation of G has been fixed, it depends only on the primary invariant decomposition of G/G', which is unique up to reordering of factors. However, calculation of a complement of $\operatorname{im} \inf$ in $H^2(G, U)$, as the image of transgression, is not canonical – it depends on the choice of a Schur complement. This is a potential source of difficulty in comparison of cocyclic matrices, or in using the decomposition of Theorem 3.6 to recognise whether a given matrix with entries in U is cocyclic over some group G. It would be useful, therefore, to have a characterisation of the elements of $\operatorname{im} \inf$ in terms of cocycle classes. A (conditional) characterisation of that sort will be presented in this section. A characterisation in terms of equivalence classes of central extensions is given in [6, p. 24, Lemma 2.1.17], but is difficult to translate into terms of cocycles.

An element ψ of $Z^2(G, U)$ will be called *almost symmetric* if $\psi(g, h) = \psi(h, g)$ whenever $[g, h] = 1$. Note that every coboundary is almost symmetric. A presentation

$$1 \to R \xrightarrow{\text{inc.}} F \xrightarrow{\text{proj.}} F/R \to 1 \qquad (6)$$

of $G \cong F/R$ will be said to satisfy (P) if $R \cap F'/[R,F]$ is generated by elements in the set of generators

$$\{[f_1, f_2][R,F] \mid f_i \in F\}$$

of $F'/[R,F]$. After calculating $H_2(G)$ in Hopf's form, inspection would hopefully reveal whether the chosen presentation of G satisfies (P).

Lemma 4.1. *Suppose G has a presentation* (6) *satisfying* (P), *and $H_2(G) \neq \{0\}$. Choose a Schur complement $S/[R,F]$. Then for all nonzero $[\psi] \in \operatorname{im} \tau_S$, ψ is not almost symmetric.*

Proof. Applying the natural isomorphism of $R \cap F'/[R,F]$ onto R/S, we see that R/S is generated by elements of the form $[f_1, f_2]S$.

Choose a normalised transversal function $v : F/R \to F$ for the presentation (6). Then $\sigma : fR \mapsto v(fR)S$ defines a normalised transversal function $\sigma : F/R \to F/S$ for (5) and we have $\mu \in Z^2(F/R, R/S)$ defined by

$$\mu(fR, gR) = v(fR)v(gR)v(fgR)^{-1}S.$$

Suppose $\phi \in \operatorname{Hom}(R/S, U)$ is nontrivial on at least one generator $[f_1^{-1}, f_2^{-1}]S$, say, of R/S. We will assume that $\phi \circ \mu$ is almost symmetric and derive a contradiction.

Since f_1R and f_2R commute, by assumption we have

$$\phi(v(f_1R)v(f_2R)v(f_1f_2R)^{-1}S) = \phi(v(f_2R)v(f_1R)v(f_2f_1R)^{-1}S),$$

and thus $\phi(v(f_1R)v(f_2R)v(f_1R)^{-1}v(f_2R)^{-1}S) = 1$. But

$$v(f_1R)v(f_2R)v(f_1R)^{-1}v(f_2R)^{-1}S = f_1f_2f_1^{-1}f_2^{-1}S$$

by centrality of R/S in F/S and so $\phi([f_1^{-1}, f_2^{-1}]S) = 1$, the required contradiction. □

Corollary 4.2. *Suppose that G has a presentation* (6) *satisfying* (P). *Then*

$$\operatorname{im}\inf = \{[\psi] \in H^2(G,U) \mid \psi \text{ is almost symmetric}\}.$$

Proof. One direction of containment is obvious. The other direction follows from Theorem 3.6 and Lemma 4.1. □

So if G has a presentation satisfying (P), each cocyclic matrix over G decomposes, not necessarily uniquely, as the coordinatewise product of a symmetric (inflation) matrix, an almost symmetric (coboundary) matrix and an asymmetric (transgression) matrix. We ask whether the dependence on (P) in Corollary 4.2 may be removed: is it true in general that each symmetric element of $Z^2(G,U)$ lies in a class $[\inf \psi]$ for some symmetric $\psi \in Z^2(G/G',U)$?

5. An example

In this section we present an application of the machinery in Section 3 to calculate representative cocyclic matrices. Specifically, we consider development over finite metacyclic groups. This is the next obvious class of groups to study after abelian groups, which are dealt with in Section 4 of [1]. Given the transparency of calculating im inf, we restrict attention to transgression matrices.

Suppose G is the (nonabelian) metacyclic group presented by the quotient F/R, where F is free on a, b and R is the normal closure in F of a^r, b^s and $b^{-1}aba$, where $(r-1)^s \equiv 1 \bmod r$. Accordingly s must be even. We will identify the elements of G with those of the transversal

$$\{a^i b^j \mid 0 \le i \le r-1, 0 \le j \le s-1\}$$

of R in F. In this case $H_2(G)$ is cyclic of known order (see [6, p. 98, Theorem 2.11.3]). If r is odd then $H_2(G) = 0$, whereas $H_2(G) = \mathbb{Z}_2$ if r is even. Consequently, from now on we assume $r = 2m$, $m > 1$ and $s = 2n$, $n \ge 1$. Furthermore, if U is finitely generated then by additivity of Hom it is sufficient to consider $U = \mathbb{Z}_2 = \{-1, 1\}$.

The first step in calculating the image of transgression in $H^2(G, \mathbb{Z}_2)$ is to determine a Schur complement for the choice of F and R above. In this step, we follow the programme laid out on p. 132 of [4] and write down a presentation of $R/[R,F]$ in terms of generators of $F/[R,F]$, and from this read off the torsion subgroup $R \cap F'/[R,F]$ of the former group.

Lemma 5.1. $[a^m, b][R,F] = (a^{2m})^{-1}(b^{-1}aba)^m[R,F]$.

Proof. An easy induction shows that

$$[a^m, b][R,F] = a^{-(m+k)}b^{-1}a^{m-k}b(b^{-1}aba)^k[R,F]$$

for all $k \ge 0$, from which the claim follows after setting $k = m$. □

Proposition 5.2.

$$R/[R,F] = \langle b^{2n}[R,F], b^{-1}aba[R,F]\rangle \times \langle [a^m, b][R,F]\rangle.$$

The first factor of this direct product is a Schur complement $S/[R,F]$ and is a free abelian group of rank 2; the second factor is $R \cap F'/[R,F] \cong \mathbb{Z}_2$.

Proof. Certainly $[a^m, b] \in R \cap F'$. We now show that $[a^m, b][R,F]$ has trivial square. Modulo $[R,F]$,

$$\begin{aligned}[a^m, b]^2 &\equiv a^{-m} \cdot b^{-1}a^m ba^{-m} \cdot b^{-1}a^m b \equiv a^{-m} \cdot b^{-1}a^m b \cdot b^{-1}a^m ba^{-m} \\ &\equiv a^{-m}b^{-1} \cdot a^{2m} \cdot ba^{-m} \equiv 1,\end{aligned}$$

using the fact that $a^{2m}[R,F]$ and $[b, a^{-m}][R,F]$ lie in the central subgroup $R/[R,F]$ of $F/[R,F]$.

Since $S/[R,F]$ has rank 2 and $R \cap F'/[R,F] \cong \mathbb{Z}_2$, we have

$$R/[R,F] \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2.$$

We must have $[a^m, b] \notin [R,F]$; otherwise, $R/[R,F]$ would be generated by fewer than three elements by Lemma 5.1. By the same reasoning, $\langle b^{2n}[R,F], b^{-1}aba[R,F]\rangle$ is genuinely a complement of $\langle [a^m,b]\,[R,F]\rangle$ in $R/\,[R,F]$. □

Thus, this class of metacyclic groups is seen to satisfy the property (P) as discussed in Section 4.

We are now in a position to calculate the image of transgression τ_S, for the choice of S made in Proposition 5.2. Note that

$$R/S = \langle [a^m,b]S\rangle = \langle a^{2m}S\rangle$$

and also that the following relations hold in F/S:

$$(bS)^{2n} = S, \qquad (aS)^{bS} = a^{-1}S.$$

A transversal function $\sigma : F/R \longrightarrow F/S$ is defined by $\sigma(a^ib^jR) = a^ib^jS$ for $0 \le i \le 2m-1$ and $0 \le j \le 2n-1$, and gives rise to $\mu \in Z^2(F/R, R/S)$ as usual. We proceed to determine the entries of a cocyclic matrix associated with $\phi \circ \mu$, where ϕ is the nonidentity element of $\mathrm{Hom}(R/S, \mathbb{Z}_2)$.

Choose $a^ib^jR, a^kb^lR \in F/R$, $0 \le i,k \le 2m-1$ and $0 \le j,l \le 2n-1$. Modulo both R and S,

$$a^ib^j \cdot a^kb^l \equiv a^{i+(-1)^jk}b^{j+l}.$$

Therefore,

$$\mu(a^ib^jR, a^kb^lR) = a^{i+(-1)^jk-\overline{i+(-1)^jk}}S,$$

where overlining denotes reduction modulo $2m$. That is, when j is even,

$$\mu(a^ib^jR, a^kb^lR) = \begin{cases} 1 & \text{if } 0 \le i+k \le 2m-1, \\ a^{2m}S & \text{otherwise,} \end{cases} \tag{7}$$

and when j is odd,

$$\mu(a^ib^jR, a^kb^lR) = \begin{cases} 1 & \text{if } i \ge k, \\ a^{2m}S & \text{if } i < k. \end{cases} \tag{8}$$

We order the elements of G by

$$a^ib^j < a^kb^l \Leftrightarrow j < l \quad \text{or} \quad \text{both} \quad j = l \quad \text{and} \quad i < k.$$

With rows and columns indexed by the elements of G under this ordering, a cocyclic matrix associated with $\phi \circ \mu$ is a $2n \times 2n$ block matrix of the form

$$\begin{pmatrix} X & X & \dots & X \\ Y & Y & \dots & Y \\ X & X & \dots & X \\ \vdots & \vdots & \vdots & \vdots \\ Y & Y & \dots & Y \end{pmatrix}, \tag{9}$$

where the blocks X and Y are $2m \times 2m$ matrices with entries ± 1 constructed according to the rules (7) and (8) respectively. In fact, it is readily seen that X is back negacyclic, and Y is that matrix obtained by writing the rows of X in reverse order. This completes our description of a representative cocyclic matrix for $\tau_S(\phi)$. A different choice of Schur complement produces a matrix that may be visibly different, but whose coordinatewise (Hadamard) product with (9) is certainly almost symmetric.

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